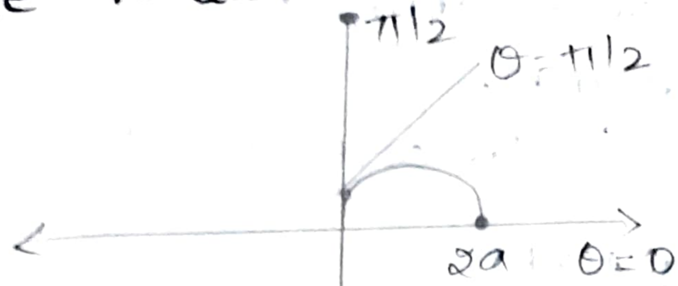
Show that $\iint_R r^2 \sin\theta \, dr \, d\theta = \frac{2a^3}{3}$, where R is the Semi-circle $r = 2a\cos\theta$ above the intial line.

Sol:-



$\theta = 0$ to $\pi/2$

$r = 0$ to $2a\cos\theta$

The given curve is $r = 2a\cos\theta$. is symmetrical about the intial line and passing through the pole.

The region of integration R above the initial line is the shaded area in the diagram.

R is varying from 0 to $2a\cos\theta$ and θ is varying from 0 to $\pi/2$.

$$\iint r^2 \sin\theta \, dr \, d\theta = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r^2 \sin\theta \, dr . d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin\theta \left(\frac{r^3}{3}\right)_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{3} \int_{\theta=0}^{\pi/2} \sin\theta \left(8a^3 \cos^3\theta\right) d\theta$$

$$= \frac{8a^3}{3} \int_{0}^{\pi/2} \sin\theta . \cos^3\theta \, d\theta$$

$$= \frac{8a^3}{3}\int_1^0 t^3(-dt)$$

$$= \frac{8a^3}{3}\int_0^1 t^3.dt$$

$$= \frac{8a^3}{3}\left(\frac{t^4}{4}\right)_0^1$$

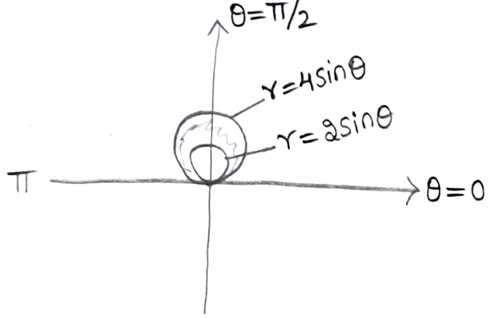
$$= \frac{\overset{2}{\cancel{8}}a^3}{3}\left(\frac{1}{\cancel{4}}\right)$$

$$= \frac{2a^3}{3}.$$

$$\iint r^2\sin\theta\, dr.\, d\theta = \frac{2a^3}{3}.$$

Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

Given region of integration R is the shaded region in the diagram.



$r = 2\sin\theta$

θ	0	$\pi/2$	π
r	0	2	0

r is varying from $r = 2\sin\theta$ to $r = 4\sin\theta$ and θ is varying from 0 to π.

$$\iint r^3 dr\, d\theta = \int_{\theta=0}^{\pi}\left[\int_{r=2\sin\theta}^{r=4\sin\theta} r^3\, dr\right]d\theta$$

$$= \int_{\theta=0}^{\pi}\left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta} d\theta.$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi}\left[(4\sin\theta)^4 - (2\sin\theta)^4\right]d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi}\left[4^4(\sin^4\theta) - 2^4(\sin^4\theta)\right]d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi} (4^4 - 2^4)\sin^4\theta \, d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi} (256-16)\sin^4\theta \, d\theta$$

$$= 60 \int_{\theta=0}^{\pi} \sin^4\theta \, d\theta$$

$$= 60 \times 2 \int_{0}^{\pi/2} \sin^4\theta \, d\theta. \qquad \left[\because \int_0^{\pi/2} \sin^n\theta \, d\theta = \frac{n-1}{n} \times \frac{n-3}{n-2} \cdots \frac{1}{2} \right.$$

$$= 120 \left[\frac{3}{4} \times \frac{1}{2} \times \pi/2 \right]$$

$$= 30 \left[\frac{3\pi}{4} \right]$$

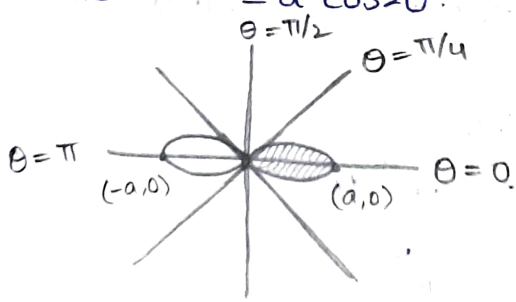
$$= \frac{45\pi}{2}.$$

$$\iint r^3 \, dr \, d\theta = \frac{45\pi}{2}.$$

Evaluate $\iint \frac{r \, dr \, d\theta}{\sqrt{a^2+r^2}}$ over the one loop of the lemiscate

$r^2 = a^2\cos 2\theta$

Given curve is $r^2 = a^2\cos 2\theta$.



θ	0	$\pi/4$	$\pi/2$	
r	$\pm a$	0	$-$	$=$

The curve $r^2 = a^2\cos 2\theta$ is symmetrical about initial line and perpendicular line and the curve is passing through the pole.

The curve intersect at the points $(a,0)$ and $(-a,0)$ at the x-axis.

$\therefore$ θ is varying from $-\pi/4$ to $+\pi/4$ and r is varying from 0 to $a\sqrt{\cos 2\theta}$.

$$\therefore \iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \int_{\theta=-\pi/4}^{\pi/4}\left[\int_{r=0}^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}}\,dr\right]d\theta$$

put $a^2+r^2=t^2$

$\cancel{2}r\,dr = \cancel{2}t\,dt$

when $r=0 \Rightarrow t=a$

$r=a\sqrt{\cos 2\theta} \Longrightarrow t=\sqrt{2}\,a\cos\theta$.

$$\int_{\theta=-\pi/4}^{\pi/4}\left[\int_{t=a}^{\sqrt{2}a\cos\theta} \frac{t\,dt}{\sqrt{t^2}}\right]d\theta$$

$$\int_{\theta=-\pi/4}^{\pi/4}\left[\int_{t=a}^{\sqrt{2}a\cos\theta} dt\right]d\theta$$

$$= \int_{\theta=-\pi/4}^{\pi/4} (t)_a^{\sqrt{2}a\cos\theta}\,d\theta$$

$$= \int_{\theta=-\pi/4}^{\pi/4} (\sqrt{2}a\cos\theta - a)\,d\theta$$

$$= \sqrt{2}a\int_{\theta=-\pi/4}^{\pi/4}\cos\theta\,d\theta - a\int_{\theta=-\pi/4}^{\pi/4} d\theta$$

$$= \sqrt{2}a(\sin\theta)_{-\pi/4}^{\pi/4} - a(\theta)_{-\pi/4}^{\pi/4}$$

$$= \sqrt{2}a\left(\sin\pi/4 - \sin(-\pi/4)\right) - a\left(\pi/4 - (-\pi/4)\right)$$

$$= \sqrt{2}a\,(2\sin\pi/4) - a(2\pi/4)$$

$$= \cancel{\sqrt{2}}a\left(2\times\frac{1}{\cancel{\sqrt{2}}}\right) - a(\pi/2)$$

$$2a - \frac{\pi a}{2}$$

$$2a(1 - \pi/4)$$

$$\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = 2a(1 - \pi/4)$$

Change of Variables in double integral

a) change of variables from cartesian to polar co-Ordin

$$\iint \frac{x^2-y^2}{x^2+y^2}\,dx\,dy.$$

put $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = |J|\,dr\,d\theta$ where $|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$= r.\,dr\,d\theta$

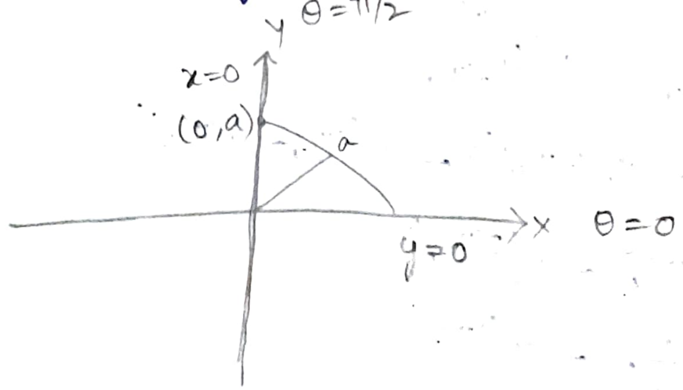
$$\iint_R F(x,y)\,dx\,dy = \iint_R F(r,\theta)\,r\,dr\,d\theta$$

Evaluate the following integral by transforming in to polar Co-Ordinates

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y.\sqrt{x^2+y^2}\,dx\,dy$$

Given region of integration is $x=0$, $x=a$, $y=0$ & $y=\sqrt{a^2-x^2}$

$x^2+y^2=a^2$



From the diagram

r is varying from 0 to a

θ is varying from 0 to $\pi/2$

put $x = r\cos\theta$, $y = r\sin\theta$ & $dx\,dy = r.\,dr\,d\theta$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y.\sqrt{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r\sin\theta\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\;r.\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\int_0^a r^3 dr\right] \sin\theta\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(\frac{r^4}{4}\right)_0^a \sin\theta.\,d\theta$$

$$= \frac{a^4}{4} \int_{\theta=0}^{\pi/2} \sin\theta\,d\theta$$

$$= \frac{a^4}{4} \left[-\cos\theta\right]_0^{\pi/2}$$

$$= -\frac{a^4}{4} \left[\cos\pi/2 - \cos 0\right]$$

$$= -\frac{a^4}{4}[0-1]$$

$$= \frac{a^4}{4}.$$

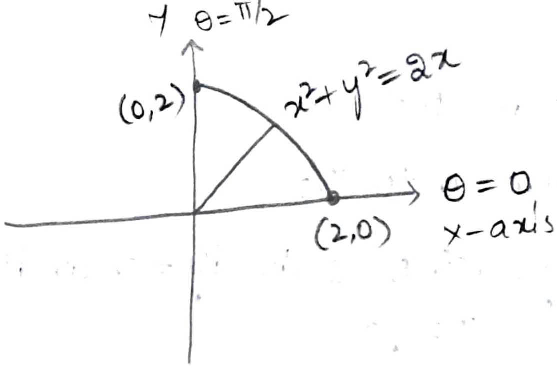
$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y.\sqrt{x^2+y^2}\,dx\,dy = \frac{a^4}{4}$$

Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$ by changing in to polar co-Ordinates.

The region of integration is $x=0$, $x=2$, $y=0$ &

$$y = \sqrt{2x-x^2}$$

$$x^2+y^2 = 2x$$



changing into polar co-ordinates

put $x = r\cos\theta$, $y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta$$

from $x^2+y^2 = 2x$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$$

$$r^{\cancel{2}} = 2\cancel{r}\cos\theta$$

$$r = 2\cos\theta$$

r is varying from 0 to $2\cos\theta$

θ is varying from 0 to $\pi/2$.

$$\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta} (r^2\cos^2\theta + r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{2\cos\theta} r^3\,dr\right]d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left(\frac{r^4}{4}\right)_0^{2\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi/2} 16\cos^4\theta\,d\theta$$

$$= 4\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$= \cancel{4}\left[\frac{3}{\cancel{4}}\times\frac{1}{2}\times\frac{\pi}{2}\right]$$

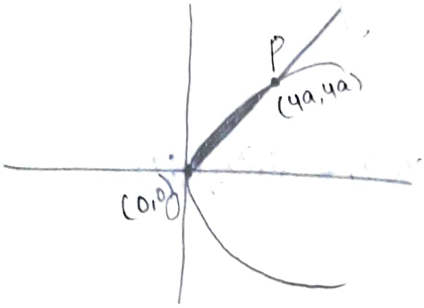
$$= \frac{3\pi}{4}$$

Show that $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = 8a^2\left(\frac{\pi}{2}-\frac{5}{3}\right)$

The region of integration is bounded by the parabola $y^2=4ax$ and the straight line $y=x$.

The region of integration is $y=0$, $y=4a$, $x=\frac{y^2}{4a}$, $x=y$

$y^2=4ax.$



put $x=r\cos\theta$, $y=r\sin\theta$; $dx\,dy=r\,dr\,d\theta$.

The limits of 'r' is $r=0$ at origin and at p the parabola

$y^2=4ax \Rightarrow r^{\cancel{2}}\sin^2\theta = 4a\cancel{r}\cos\theta$

$$r=\frac{4a\cos\theta}{\sin^2\theta}.$$

$\therefore r$ is varying from 0 to $\frac{4a\cos\theta}{\sin^2\theta}$.

The slope of the line $y=x$ is $m=1$.

$\tan\theta = 1$

$\theta=\pi/4$

$\therefore \theta$ is varying from $\pi/4$ to $\pi/2$.

Now, $x^2-y^2 = r^2\cos^2\theta - r^2\sin^2\theta$

$= r^2(\cos^2\theta-\sin^2\theta)$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2$

$$\frac{x^2-y^2}{x^2+y^2} = \frac{\cancel{r^2}(\cos^2\theta-\sin^2\theta)}{\cancel{r^2}}$$

$$\frac{x^2-y^2}{x^2+y^2} = \cos^2\theta - \sin^2\theta$$

$$\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} (\cos^2\theta-\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2}\left[\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} r\,dr\right](\cos^2\theta-\sin^2\theta)\,d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2}\left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}}(\cos^2\theta-\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\theta=\pi/4}^{\pi/2} 16a^2\,\frac{\cos^2\theta}{\sin^4\theta}(\cos^2\theta-\sin^2\theta)\,d\theta$$

$$= 8a^2\int_{\theta=\pi/4}^{\pi/2}\frac{\cos^4\theta}{\sin^4\theta}\,d\theta - 8a^2\int_{\theta=\pi/4}^{\pi/2}\frac{\cos^2\theta}{\sin^2\theta}$$

$$= 8a^2\int_{\theta=\pi/4}^{\pi/2}\cot^4\theta\,d\theta - 8a^2\int_{\theta=0}^{\pi/4}\cot^2\theta\,d\theta$$

$$= 8a^2\left[\frac{1}{3}-\frac{1}{1}+\frac{\pi}{4}\right] - 8a^2\left[\frac{1}{1}-\frac{\pi}{4}\right]$$

$$= 8a^2\left[\frac{1}{3}+\frac{2\pi}{4}-2\right]$$

$$= 8a^2\left[\frac{1}{3}+\frac{\pi}{2}-2\right]$$

$$= 8a^2\left[\frac{2+3\pi-12}{6}\right]$$

$$= 8a^2\left[\frac{3\pi-10}{6}\right]$$

$$= 8a^2\left[\frac{\pi}{2}-\frac{5}{3}\right]$$

Hence proved.

By changing in to polar Co-ordinates. Evaluate $\iint \frac{x^2y^2}{x^2+y^2}\, dx\, dy$ over the annular region between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$

Sol:- By changing in to polar co-ordinates

put $x=r\cos\theta$, $y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

$x^2y^2 = r^2\cos^2\theta \cdot r^2\sin^2\theta$

$= r^4\sin^2\theta \cdot \cos^2\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2$

$$\frac{x^2y^2}{x^2+y^2} = \frac{r^{\cancel{4}\,2}\sin^2\theta\cos^2\theta}{\cancel{r^2}}$$

$$\frac{x^2y^2}{x^2+y^2} = r^2\sin^2\theta\cos^2\theta.$$

θ is varying from 0 to 2π.

r is varying from a to b.

$$\iint \frac{x^2y^2}{x^2+y^2}\, dx\, dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^2\sin^2\theta\cos^2\theta\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{r^4}{4}\right)_a^b \sin^2\theta \cdot \cos^2\theta\, d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{2\pi} (b^4-a^4)\sin^2\theta \cdot \cos^2\theta\, d\theta$$

$$= \frac{b^4-a^4}{16}\int_{\theta=0}^{2\pi} 4\sin^2\theta \cdot \cos^2\theta\, d\theta$$

$$= \frac{b^4-a^4}{16} \int_{\theta=0}^{2\pi} (2\sin\theta\cos\theta)^2 \, d\theta$$

$$= \frac{b^4-a^4}{16} \int_{\theta=0}^{2\pi} (\sin 2\theta)^2 \, d\theta.$$

$$= \frac{b^4-a^4}{16} \int_{\theta=0}^{2\pi} \left[\frac{1-\cos 4\theta}{2}\right] d\theta.$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 4\theta = 1 - 2\sin^2 2\theta$

$2\sin^2 2\theta = 1 - \cos 4\theta.$

$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$

$$= \frac{b^4-a^4}{16} \left[\int_0^{2\pi} \frac{1}{2} d\theta - \int_{\theta=0}^{2\pi} \frac{\cos 4\theta}{2} d\theta \right]$$

$$= \frac{b^4-a^4}{32} \left[(\theta)_0^{2\pi} - \left(\frac{\sin 4\theta}{4}\right)_0^{2\pi} \right]$$

$$= \frac{b^4-a^4}{\cancel{32}\, 16} \times \cancel{2}\pi$$

$$= \frac{\pi(b^4-a^4)}{16}.$$

$$\iint \frac{x^2y^2}{x^2+y^2} \, dx\, dy = \frac{\pi(b^4-a^4)}{16}.$$

Change of order of Integration :- It implies that the change of limits of integration.

i.e $$\int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} F(x,y)\, dy\, dx = \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} F(x,y)\, dx\, dy.$$

$x \rightarrow$ constant limits

$y \rightarrow$ variable limits

$y \rightarrow$ constant limits

$x \rightarrow$ Variable limits.

procedure :-

i) First identify the variables for the limits.

ii) Draw a rough sketch of the given region of integration.

iii) If we are evaluating the integral with respect to y then take a vertical strip that is a line parallel to y-axis, otherwise take a horizontal strip that is a line parallel to x-axis.

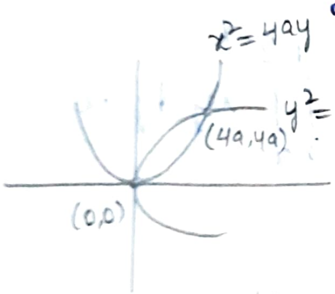
iv) Identify the limits and evaluate the double integral with new order of integration.

Change the order of integration and evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

Sol:- The region of integration is

$x=0,\ x=4a,\ y=x^2/4a \ \&\ y=2\sqrt{ax}$

$x^2=4ay \qquad y^2=4ax$



when $x=0,\ y=0$

$x=4a,\ y=4a$

point of intersection is $(0,0)$ and $(4a,4a)$

for a fixed y, x is varying from $y^2/4a$ to $2\sqrt{ay}$ and then y is varying from 0 to 4a.

$$\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy$$

$$= \int_{y=0}^{4a} (x)_{y^2/4a}^{2\sqrt{ay}}\,dy$$

$$= \int_{y=0}^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$2\sqrt{a}\int_0^{4a}\sqrt{y}\,dy - \frac{1}{4a}\int_0^{4a} y^2\,dy$$

$$2\sqrt{a}\left(\frac{y^{3/2}}{3/2}\right)_0^{4a} - \frac{1}{4a}\left(\frac{y^3}{3}\right)_0^{4a}$$

$$2\sqrt{a}\times\frac{2}{3}(4a)^{3/2} - \frac{1}{12a}(4a)^3$$

$$\frac{4\sqrt{a}}{3}(2^2)^{3/2}\, a\sqrt{a} - \frac{1}{12\cancel{a}}\,64a^{\cancel{3}}$$

$$\frac{4a^2}{3}\times 8 - \frac{16}{3}a^2$$

$$\frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$ //

Change the order of integration $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the double integrals.

The correct order of the integration is $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$

The region of integration

$x=0,\ x=1,\ y=x^2$ and $y=2-x$ i.e $x+y=2$.

from $x+y=2$

$x=2$ (on x-axis)

$y=2$ (on y-axis)

The ~~integration~~ intersecting point of the two curves $y=x^2$ & $y=2-x$ is $x^2=2-x$

$$x^2+x-2=0$$

$$x^2+2x-x-2=0$$

$$x(x+2)-1(x+2)=0$$

$$(x-1)(x+2)=0$$

$$x=1,\ x=-2.$$

$$(x,y)=(1,1)$$

By chaning the order of integration the total area can be divided in to two regions.

Area OABO = Area OACO + Area ABCA

for the area OACO

for a fixed y, x is varying from 0 to $\sqrt{y}$ and then y is varying from 0 to 1

for the area ABCA.

for a fixed y. x is varying from 0 to 2-y and then y is varying from 1 to 2.

$$\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_{y=1}^{2} y\left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 y(4+y^2-4y)\,dy$$

$$= \frac{1}{2}\left[\frac{1}{3}\right] + \frac{1}{2}\left[4\cdot\frac{y^2}{2} + \frac{y^4}{4} - 4\cdot\frac{y^3}{3}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[2(2^2-1) + \frac{1}{4}(2^4-1^4) - \frac{4}{3}(2^3-1^3)\right].$$

$$= \frac{1}{6} + \frac{1}{2}\left[2(4-1) + \frac{1}{4}(16-1) - \frac{4}{3}(8-1)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left(6 + \frac{15}{4} - \frac{28}{3}\right)$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{72+45-112}{12}\right)$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{5}{12}\right)$$

$$= \frac{1}{6} + \frac{5}{24}$$

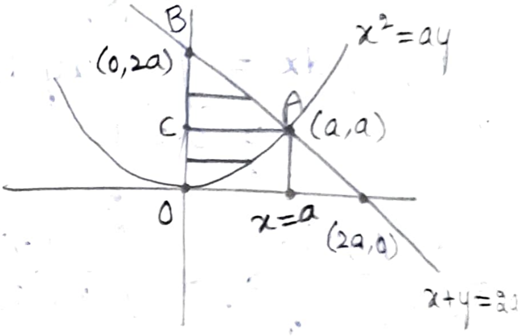
$$= \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8}.$$

$$\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy = \frac{3}{8}$$

Change the order of integration and evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy^2\, dy\, dx$

The region of integration

$x=0,\ x=a$



$y = x^2/a$ & $y=2a-x$

$x^2 = ay$ & $x+y=2a$

$$x + \frac{x^2}{a} = 2a$$

$$ax + x^2 = 2a^2$$

$$x^2 + ax - 2a^2 = 0.$$

$$x^2 + 2ax - ax - 2a^2 = 0$$

$$x(x+2a) - a(x+2a) = 0$$

$$(x-a)(x+2a) = 0$$

$$x = a,\ x = -2a$$

By the changing the order of integration

Area of OABO = OACO + ABCA

for a fixed y, x is varying from 0 to $\sqrt{ay}$ and y is varying from 0 to a

for the region ABCA, x is varying from 0 to $2a-y$ and y is varying from 0 to $2a$.

$$\int_0^a \int_{x^2/2a}^{2a-x} xy^2\, dy\, dx = \int_{y=0}^{a} \int_{x=0}^{\sqrt{ay}} xy^2\, dx\, dy + \int_{y=a}^{2a} \int_{x=0}^{2a-y} xy^2\, dx\, dy$$

$$= \int_{y=0}^{a}\left[\int_{x=0}^{\sqrt{ay}} x\,dx\right]y^2\,dy + \int_{y=a}^{2a}\left[\int_{0}^{2a-y} x\,dx\right]y^2\,dy$$

$$= \int_{y=0}^{a}\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} y^2\,dy + \int_{y=a}^{2a}\left[\frac{x^2}{2}\right]_0^{2a-y} y^2\,dy$$

$$= \frac{1}{2}\int_{y=0}^{a}(ay)y^2\,dy + \frac{1}{2}\int_{y=a}^{2a}(2a-y)^2y^2\,dy$$

$$= \frac{1}{2}\int_{y=0}^{a} ay^3\,dy + \frac{1}{2}\int_{y=a}^{2a}(4a^2+y^2-4ay)y^2\,dy$$

$$= \frac{a}{2}\int_{y=0}^{a} y^3\,dy + \frac{1}{2}\int_{y=a}^{2a} 4a^2y^2+y^4-4ay^3\,dy$$

$$= \frac{a}{2}\int_{y=0}^{a} y^3\,dy + \frac{1}{2}\left[4a^2\int_{y=a}^{2a} y^2\,dy + \int_{y=a}^{2a} y^4\,dy - 4a\int_{y=a}^{2a} y^3\,dy\right]$$

$$= \frac{a}{2}\left(\frac{y^4}{4}\right)_0^a + \frac{1}{2}\left[4a^2\left(\frac{y^3}{3}\right)_a^{2a} + \left(\frac{y^5}{5}\right)_a^{2a} - 4a\left(\frac{y^4}{4}\right)_a^{2a}\right]$$

$$= \frac{a}{8}(a^4) + \frac{1}{2}\left[\frac{4a^2}{3}(8a^3-a^3) + \frac{1}{5}(32a^5-a^5) - \frac{4a}{4}(16a^4-a^4)\right]$$

$$= \frac{a}{8}(a^4) + \frac{1}{2}\left[\frac{4a^2}{3}(7a^3) + \frac{1}{5}(31a^5) - a(15a^4)\right]$$

$$= \frac{a^5}{8} + \frac{1}{2}\left[\frac{28a^5}{3} + \frac{31a^5}{5} - 15a^5\right]$$

$$= \frac{a^5}{8} + \frac{1}{2}\left[\frac{140a^5+233a^5-225a^5}{15}\right]$$

$$= \frac{a^5}{8} + \frac{1}{2}\left[\frac{8a^5}{15}\right]$$

$$= \frac{a^5}{8} + \frac{8a^5}{30} = \frac{15a^5+32a^5}{120}$$

$$\text{Area} = \frac{47a^5}{120} \text{ sq. units.}$$

Change the order of integration and hence evaluate

$$\int_0^3 \int_1^{\sqrt{4-y^2}} (x+y)\, dx\, dy$$

The region of integration

$y=0,\ y=3,\ x=1,\ x=\sqrt{4-y^2}$.

i.e $x^2+y^2=4$.

put $y=0$ $x=\pm 2$.

$x=0$ $y=\pm 2$.

Diagram

for a fixed x, y is varying from 0 to $\sqrt{4-x^2}$ and then x is varying from 1 to 2.

$$\int_0^3 \int_1^{\sqrt{4-y^2}} (x+y)\, dx\, dy = \int_{x=1}^{2} \int_{y=0}^{\sqrt{4-x^2}} (x+y)\, dy \cdot dx.$$

$$= \int_{x=1}^{2} \left[x(y)_0^{\sqrt{4-x^2}} + \left(\frac{y^2}{2}\right)_0^{\sqrt{4-x^2}} \right] dx$$

$$= \int_{x=1}^{2} x\sqrt{4-x^2}\, dx + \frac{1}{2}\int_1^2 (4-x^2)\, dx$$

Put $4-x^2=t$

$-2x\,dx=dt$

$x\,dx=-dt/2$.

when $x=1 \Rightarrow t=3$.

$x=2 \Rightarrow t=0$

$$= \int_{t=3}^{0} \sqrt{t}\left(\frac{-dt}{2}\right) + \frac{1}{2}\int_1^2 4\,dx - \frac{1}{2}\int_1^2 x^2\,dx$$

$$= \int_0^3 \sqrt{t}\, dt + \frac{1}{2}\int_1^2 4\, dx - \frac{1}{2}\int_1^2 x^2\, dx$$

$$= \left(\frac{t^{3/2}}{3/2}\right)_0^3 + \frac{1}{2} \times 4 (x)_1^2 - \frac{1}{2}\left(\frac{x^3}{3}\right)_1^2$$

$$= \frac{2}{3}(3)^{3/2} + 2(2-1) - \frac{1}{6}(8-1)$$

$$= \frac{2}{\cancel{3}} \times \cancel{3}\sqrt{3} + 2 - \frac{7}{6}$$

$$= 2\sqrt{3} + \frac{5}{6}.$$

Area's

Formula's :-

Cartesian Co-ordinates :-

1) Area $A = \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} dy \cdot dx$

2) Area $A = \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} dx\, dy.$

Polar Co-ordinates.

$$\text{Area} = \int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{f_2(\theta)} r\, dr\, d\theta$$

Find the area of a circle using double integral.

Sol :- Let the circle equation be $x^2+y^2=a^2$

From the diagram.

for a fixed x, y is varying from 0 to $\sqrt{a^2-x^2}$ and then x is varying from 0 to a.

Required area $= 4 \times$ Area of OAB.

$$= 4 \int \int_{}^{a} dx\, dy.$$

$$= 4 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} dy \cdot dx$$

$$= 4 \int_{0}^{a} (y)_{0}^{\sqrt{a^2-x^2}} dx$$

$$= 4 \int_{0}^{a} \sqrt{a^2-x^2}\, dx$$

$$= 4 \left[\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}(x/a) \right]_{0}^{a}$$

$$= 4 \left[\frac{a^2}{2} \sin^{-1}(1) - \frac{a^2}{2} \sin^{-1}(0) \right]$$

$$= \cancel{4}^{2} \left(\frac{a^2}{\cancel{2}} \right) \pi/\cancel{2} = \pi a^2 \text{ sq. units}$$

Find the area enclosed by the parabola $x^2 = y$ and $y^2 = x$.

Sol:- Given parabola's are

$x^2 = y$ — ① and $y^2 = x$ — ②

Solving ① & ② we get

from ②

$$(x^2)^2 = x$$

$$x^4 - x = 0$$

$$x(x^3 - 1) = 0$$

$x=0$ and $x=1$

point of intersection is $(0,0)$ & $(1,1)$

for a fixed y, x is varying from y^2 to $\sqrt{y}$ and then y is varying from 0 to 1

$$\text{Area} = \iint dx\, dy$$

$$= \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} dx\, dy$$

$$= \int_{y=0}^{1} (x)_{y^2}^{\sqrt{y}}\, dy$$

$$= \int_{0}^{1} (\sqrt{y} - y^2)\, dy$$

$$= \left(\frac{y^{3/2}}{3/2} - \frac{y^3}{3}\right)_0^1$$

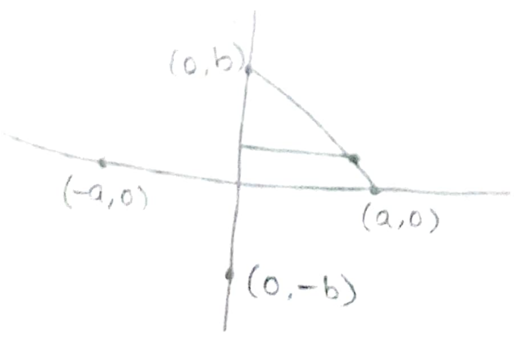
$$= \frac{2}{3}\left(1^{3/2}\right) - \frac{1}{3}(1)$$

$$= \frac{2}{3} - \frac{1}{3} = 1/3$$

Area $= \frac{1}{3}$ sq. units.

Find the area of a plane in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol:- Given ellipse equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$x^2 = \frac{a^2}{b^2}(b^2 - y^2)$$

$$x = \frac{a}{b}\sqrt{b^2 - y^2}$$

x is varying from 0 to $\frac{a}{b}\sqrt{b^2-y^2}$.

y is varying from 0 to b.

$$\text{Area} = \iint dx\, dy.$$

$$= \int_{y=0}^{b} \int_{x=0}^{a/b\sqrt{b^2-y^2}} dx\, dy$$

$$= \int_0^b (x)_0^{a/b\sqrt{b^2-y^2}}\, dy.$$

$$= \int_0^b \frac{a}{b}\sqrt{b^2-y^2}\, dy.$$

$$= \frac{a}{b}\left[\frac{y}{2}\sqrt{b^2-y^2} + \frac{b^2}{2}\sin^{-1}(y/b)\right]_0^b$$

$$= \frac{a}{\cancel{b}}\left[\frac{b^{\cancel{2}}}{2} \times \pi/2\right] = \frac{\pi ab}{4}$$

$$\text{Area} = \frac{\pi ab}{4} \text{ sq.units.}$$

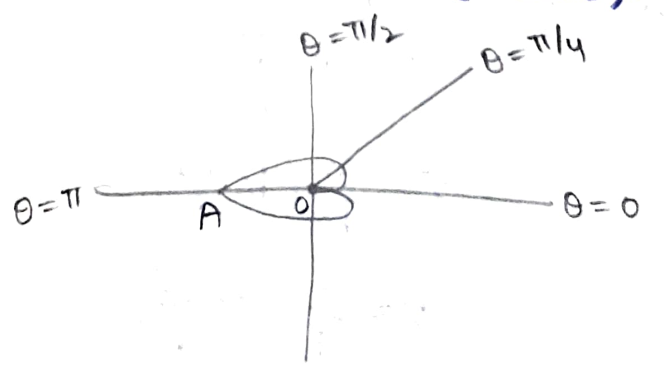
Note :- Area of the ellipse $= 4 \times$ area of the first quadrant

$$= \cancel{4} \times \frac{\pi ab}{\cancel{4}}$$

$$= \pi ab \text{ sq. units.}$$

S. Find the area of the cordiod $r = a(1-\cos\theta)$

Sol :- Given that $r = a(1-\cos\theta)$.



θ	0	$\pi/4$	$\pi/2$	π
r	0	$a(1-\frac{1}{\sqrt{2}})$	a	$2a$

The cordiod $r=a(1-\cos\theta)$ is symmetrical about the initial line. $\theta=0$ and passing through the pole.

θ is varying from 0 to π and

r is varying from 0 to $a(1-\cos\theta)$.

required area $= 2 \times$ Area of OA.

$$= 2\iint r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta.$$

$$= 2\int_{\theta=0}^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta.$$

$$= \int_{\theta=0}^{\pi} a^2(1-\cos\theta)^2\,d\theta$$

$$= a^2\int_{\theta=0}^{\pi}(2\sin^2\theta/2)^2\,d\theta.$$

$$= a^2\int_{\theta=0}^{\pi} 4\sin^4\theta/2\,d\theta.$$

put $\theta/2 = t$

$\theta = 2t$

$d\theta = 2dt$

$\theta=0 \Rightarrow t=0$

$\theta=\pi \Rightarrow t=\pi/2$

$$= 4a^2\int_0^{\pi/2}\sin^4 t\,.\,2dt$$

$$= 8a^2\int_0^{\pi/2}\sin^4 t\,.\,dt$$

$$= 8a^2\left[\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right] = \frac{3a^2\pi}{2}\ \text{sq. units.}$$

Find the area of the curve inside the circle $r = a\sin\theta$ and outside the cordiod $r = a(1-\cos\theta)$.

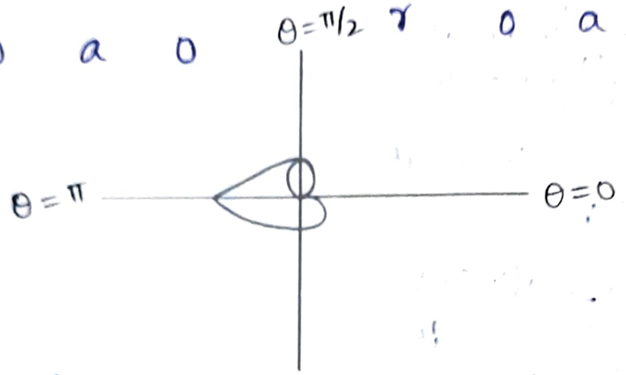
Given that $r = a\sin\theta$ and $r = a(1-\cos\theta)$

$r = a\sin\theta$

θ	0	$\pi/2$	π
r	0	a	0

$r = a(1-\cos\theta)$

θ	0	$\pi/2$	π
r	0	a	$2a$



Area $= \iint r\,dr\,d\theta$

θ is varying from 0 to $\pi/2$

r is varying from $a(1-\cos\theta)$ to $a\sin\theta$.

$$\text{Area} = \int_{\theta=0}^{\pi/2}\int_{r=a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2\sin^2\theta - a^2(1-\cos\theta)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2\sin^2\theta - a^2(1+\cos^2\theta - 2\cos\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2\sin^2\theta - a^2 - a^2\cos^2\theta + 2a^2\cos\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[a^2[\sin^2\theta - \cos^2\theta) + 2a^2\cos\theta - a^2\right]d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[-a^2(\cos^2\theta - \sin^2\theta) + 2a^2\cos\theta - a^2\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[-a^2\cos 2\theta + 2a^2\cos\theta - a^2\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[-a^2(\cos 2\theta) + 2a^2\cos\theta - a^2\right] d\theta$$

$$= \frac{1}{2}\left[-a^2\left(\frac{\sin 2\theta}{2}\right)_0^{\pi/2} + 2a^2(\sin\theta)_0^{\pi/2} - a^2(\theta)_0^{\pi/2}\right]$$

$$= \frac{1}{2}\left[0 + 2a^2(1-0) - a^2(\pi/2)\right]$$

$$= \frac{1}{2}\left[2a^2 - a^2\frac{\pi}{2}\right]$$

$$= \frac{a^2}{2}\left[2 - \frac{\pi}{2}\right]$$

$$= a^2 - \frac{\pi a^2}{4}$$

$$= a^2\left(1 - \frac{\pi}{4}\right)$$

Area $= a^2\left(1 - \frac{\pi}{4}\right)$ sq. units.

Triple Integrals.

Evaluate $\int_0^1\int_1^2\int_2^3 xyz\,dx\,dy\,dz$

Sol:- Given that $\int_0^1\int_1^2\int_2^3 xyz\,dx\,dy\,dz$

$$= \int_0^1\int_1^2 \left(\frac{x^2}{2}\right)_2^3 yz\,dy\,dz$$

$$= \int_0^1\int_1^2 \left(\frac{9}{2} - \frac{4}{2}\right) yz\,dy\,dz$$

$$5/2 \int_0^1 \left(\frac{y^2}{2}\right)_1^2 z\, dz.$$

$$5/2 \int_0^1 \left(\frac{4}{2}-\frac{1}{2}\right) z\, dz$$

$$\frac{5}{2} \times \frac{3}{2} \int_0^1 z\, dz$$

$$\frac{5}{2} \times \frac{3}{2} \left(\frac{z^2}{2}\right)_0^1$$

$$= \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} = \frac{15}{8}$$

$$\int_0^1 \int_1^2 \int_2^3 xyz\, dx\, dy\, dz = \frac{15}{8}$$

Evaluate $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz.$

Given that $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$

$$= \int_{-1}^{1} \int_{0}^{z} \left[\int_{x-z}^{x+z} (x+y+z)\, dy \right] dx\, dz.$$

$$= \int_{-1}^{1} \int_{0}^{z} \left[xy + \frac{y^2}{2} + zy \right]_{x-z}^{x+z} dx\, dz.$$

$$= \int_{-1}^{1} \int_{0}^{z} \left[(x+z)(\cancel{x}+z-\cancel{x}+z) + \left[\frac{(x+z)^2}{2} - \frac{(x-z)^2}{2} \right] \right] dx\, dz$$

$$= \int_{-1}^{1} \int_{0}^{z} \left((x+z)(2z) + \frac{4xz}{2} \right) dx\, dz$$

$$= \int_{-1}^{1}\int_{0}^{z} (2xz + 2z^2 + 2xz)\, dx\, dz$$

$$= \int_{-1}^{1}\int_{0}^{z} (4xz + 2z^2)\, dx\, dz.$$

$$= \int_{-1}^{1} \left[4z\left(\frac{x^2}{2}\right)_0^z + 2z^2(x)_0^z\right] dx z.$$

$$= \int_{-1}^{1} [2z.z^2 + 2z^2.z]\, dz.$$

$$= \int_{-1}^{1} 4z^3\, dz$$

$$= 4\left(\frac{z^4}{4}\right)_{-1}^{1}$$

$$= (1)^4 - (-1)^4$$

$$= 1 - 1 = 0$$

Evaluate $\iiint xy^2z\, dx\, dy\, dz$ taken through the positive quadrant of the sphere.

Sol:- Given sphere is $x^2+y^2+z^2=a^2$

$$z^2 = a^2 - x^2 - y^2$$

$$z = \sqrt{a^2 - x^2 - y^2}$$

z is varying from 0 to $\sqrt{a^2-x^2-y^2}$

from $x^2+y^2=a^2$.

$$y = \pm\sqrt{a^2-x^2}$$

y is varying from 0 to $\sqrt{a^2-x^2}$

from $x^2=a^2$

$$x = \pm a$$

x is varying from 0 to a.

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\left[x+y+(1-x-y)+1\right]^{-2} - (x+y+1)^{-2}\Big]\,dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\left(2^{-2}-(x+y+1)^{-2}\right)dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\frac{1}{4}\,dy\,dx + \frac{1}{2}\int_0^1\int_0^{1-x}(x+y+1)^{-2}\,dy.\,dx$$

$$= \frac{-1}{8}\int_0^1 (y)_0^{1-x}\,dx + \frac{1}{2}\int_0^1\left[\frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x}dx$$

$$= \frac{-1}{8}\int_0^1(1-x)\,dx - \frac{1}{2}\left[\int_0^1\left[(\cancel{x}+1-\cancel{x}+1)^{-1}-(1+x)^{-1}\right]dx\right.$$

$$= \frac{-1}{8}\int_0^1(1-x)\,dx - \frac{1}{2}\left[\int_0^1\left[2^{-1}-(1+x)^{-1}\right]dx\right.$$

$$= \frac{-1}{8}\int_0^1(1-x)\,dx - \frac{1}{2}\int_0^1\frac{1}{2}\,dx + \frac{1}{2}\int_0^1\frac{1}{1+x}\,dx$$

$$= \frac{-1}{8}\left[x-\frac{x^2}{2}\right]_0^1 - \frac{1}{4}(x)_0^1 + \frac{1}{2}\log(1+x)_0^1$$

$$= \frac{-1}{8}\left[1-\frac{1}{2}\right] - \frac{1}{4}(1) + \frac{1}{2}\log 2$$

$$= \frac{-1}{8}\left(\frac{1}{2}\right) - \frac{1}{4} + \frac{1}{2}\log 2$$

$$= \frac{-1}{16} - \frac{1}{4} + \frac{1}{2}\log 2$$

$$= \frac{-5}{16} + \frac{1}{2}\log 2$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}.$$

Volume :-

To find the volume of the solid is $\iiint_V dv = \iiint_V dx\,dy\,dz$.

Find the volume of the tetrahedron bounded by the planes $x=0,\ y=0,\ z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Required volume is $\iiint dx\,dy\,dz$

on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

z is varying from 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

y is varying from 0 to $b\left(1-x/a\right)$

x is varying from 0 to a.

$$\iiint_V dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)}\int_{z=0}^{c(1-x/a-y/b)} dz.\,dy.\,dx$$

$$=\int_0^a\int_0^{b(1-x/a)} (z)_0^{c(1-x/a-y/b)}\,dy\,dx$$

$$=\int_0^a\int_0^{b(1-x/a)} c\left(1-x/a-y/b\right) dy.\,dx$$

$$=c\int_0^a\left(\left(1-\frac{x}{a}\right)y-\frac{1}{b}\cdot\frac{y^2}{2}\right)_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$=c\int_0^a\left[\left(1-x/a\right)b\left(1-x/a\right)-\frac{1}{2b}\cdot b^2\left(1-x/a\right)^2\right]dx$$

$$=c\int_0^a\left(1-x/a\right)^2\left(b-\frac{b}{2}\right)dx$$

$$\frac{bc}{2}\int_0^a (1-x/a)^2\,dx$$

$$= \frac{bc}{2}\int_0^a \left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)dx$$

$$= \frac{bc}{2}\left[x+\frac{x^3}{3a^2}-\frac{\cancel{2}x^2}{\cancel{2}a}\right]_0^a$$

$$= \frac{bc}{2}\left(a+\frac{a^{\cancel{3}}}{3\cancel{a^2}}-\frac{a^{\cancel{2}}}{\cancel{a}}\right)$$

$$= \frac{bc}{2}\left(\cancel{a}+\frac{a}{3}-\cancel{a}\right)$$

$$= \frac{abc}{6} \text{ cubic units.}$$

Find the volume bounded by the xy plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$

Given that, on the surface $z=0$. (since on xy plane)

$x^2+y^2=1$ and $x+y+z=3$.

from $x+y+z=3$

$z=3-x-y$.

z is varying from 0 to $3-x-y$.

from $x^2+y^2=1$

$y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$

y is varying from $-\sqrt{1-x^2}$ to $+\sqrt{1-x^2}$.

x is varying from -1 to $+1$.

The required volume $V=\iiint dx\,dy\,dz$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{3-x-y} dz \cdot dy \cdot dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (z)_0^{3-x-y} \, dy \, dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3-x-y) \, dy \cdot dx$$

$$= \int_{-1}^{1} \left[(3-x)y - \frac{y^2}{2} \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} (3-x)\left(\sqrt{1-x^2} + \sqrt{1-x^2}\right) - \left[\frac{1}{2}(1-x^2) - \frac{1}{2}(1-x^2)\right] dx$$

$$= \int_{-1}^{1} 2(3-x)\sqrt{1-x^2} \, dx$$

$$= \int_{-1}^{1} (6-2x)\sqrt{1-x^2} \, dx$$

$$= \int_{-1}^{1} 6\sqrt{1-x^2} \, dx - \int_{-1}^{1} x\sqrt{1-x^2} \, dx$$

$$= 2\int_{0}^{1} 6\sqrt{1-x^2} \, dx - 0 \qquad \left[\begin{array}{lll} \sqrt{1-x^2} & \text{is} & \text{even function} \\ x & \text{is} & \text{odd function} \end{array} \right]$$

$$= 12 \int_{0}^{1} \sqrt{1-x^2} \, dx$$

$$= 12 \left[\frac{x}{1}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x) \right]_0^1$$

$$= 12 \left[\frac{1}{2}\sin^{-1}(1) - \frac{1}{2}\sin^{-1}(0) \right]$$

$$= \cancel{12}^{3} \times \frac{1}{\cancel{2}} \times \frac{\pi}{\cancel{2}}$$

$$= 3\pi \text{ cubic units}$$

A double or triple integral is known as multiple integral is an extension of definite integral of a fu[nction] of single variable to a function of two or three va[riables].

* The multiple integrals are useful in evaluating area volume in plane and solid Regions.

Double integrals :-

Consider a region R in the xy plane bounde[d by] one or more curves. Let $f(x,y)$ be a function defi[ned] at all points of R, then the double integral of $f(x,y)$ over the region R is denoted by $\iint_R f(x,y)\, dR$ (or) $\iint_R f(x,y)\, dx\, dy$.

Iterated integral :- An expression of the form

$$\int_a^b \int_{y=f(x_1)}^{y=f(x_2)} f(x,y)\, dy\, dx \quad \text{(or)} \quad \int_a^b \int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\, dx\, dy$$

is called an iterated integrals.

Evaluation of double integrals :-

If all the core limits of integration are constants then the double integral can be evaluated in either way i.e we first integrate with respect to x and then with respect to y. (OR) we first integrate with respect to y and then with respect to x.

i.e If both limits are constants then we take the order of appearance of the differential co-efficient $(dx \cdot dy)$.

* If the limits for one variable is function of other and constants for other variable, then we have to integrate the variable with respect to variable for which limits are functions of the other, then the variable with constant limits.

i.e $$\int_a^b \int_{y=f(x_1)}^{y=f(x_2)} F(x,y)\, dy \cdot dx \quad \text{(or)} \quad \int_a^b \int_{x=f(y_1)}^{x=f(y_2)} F(x,y) = dx \cdot dy.$$

* While evaluating double integral or multiple integral with respect to one variable, the other variables are treated as constants.

Evaluate $\int_0^3 \int_1^2 xy(1+x+y)\, dy \cdot dx$.

Here all the four limits are constant. So the double integral can be evaluated to take the order of preference of the differential co-efficient $dy \cdot dx$ i.e integrating first w.r. to y and then w.r. to x

Given,

$$\int_0^3 \int_1^2 xy(1+x+y)\, dy \cdot dx$$

$$\int_{x=0}^3 \int_{y=1}^2 (xy + x^2y + xy^2) dy \cdot dx$$

$$\int_{x=0}^3 \left[\int_{y=1}^2 (xy + x^2y + xy^2) dy \right] dx$$

$$\int_{x=0}^3 \left[\int_{y=1}^2 x \cdot y\, dy + \int_{y=1}^2 x^2y\, dy + \int_{y=1}^2 xy^2\, dy \right] dx$$

$$\int_{x=0}^{3}\left[x\left(\frac{y^2}{2}\right)_1^2 + x^2\left(\frac{y^2}{2}\right)_1^2 + x\left(\frac{y^3}{3}\right)_1^2\right]dx$$

$$\int_{x=0}^{3}\left[\frac{x}{2}[4-1] + \frac{x^2}{2}[4-1] + \frac{x}{3}[8-1]\right]dx$$

$$\frac{3}{2}\int_0^3 x\,dx + \frac{3}{2}\int_0^3 x^2\,dx + \frac{7}{3}\int_0^3 x\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}\right]_0^3 + \frac{3}{2}\left[\frac{x^3}{3}\right]_0^3 + \frac{7}{3}\left(\frac{x^2}{2}\right)_0^3$$

$$= \frac{3}{4}[9-0] + \frac{3}{6}[27-0] + \frac{7}{6}(9-0)$$

$$= \frac{3}{4}\times 9 + \frac{\cancel{3}}{\cancel{6}_2}\times 27 + \frac{7}{\cancel{6}_2}\times \cancel{9}\,3$$

$$= \frac{27}{4} + \frac{27}{2} + \frac{21}{2}$$

$$= \frac{27+54+42}{4}$$

$$= \frac{123}{4}$$

$$\int_{x=0}^{3}\int_{y=1}^{2} xy(1+x+y)\,dy\,dx = \frac{123}{4}$$

Evaluate $\int_0^2\int_0^x y.\,dy.\,dx$

Sol :- The given integral is $\int_{x=0}^{2}\left[\int_{y=0}^{x} y\,dy\right]dx$

$$= \int_{x=0}^{2} \left(\frac{y^2}{2}\right)_0^x dx$$

$$= \frac{1}{2}\int_{x=0}^{2} (x^2 - 0)\,dx$$

$$= \frac{1}{2}\int_0^2 x^2\,dx$$

$$= \frac{1}{2}\int_0^2 x^2\,dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3}\right)_0^2$$

$$= \frac{1}{6}(8-0) = 8/6 = 4/3.$$

$$\int_0^2\int_0^x y.\,dy\,dx = 4/3.$$

Evaluate $\int_0^a \int_0^b (x^2+y^2)\,dy\,dx$

The given integral is $\int_{x=0}^{a}\int_{y=0}^{b} (x^2+y^2)\,dy.dx$

$$= \int_{x=0}^{a}\left[\int_0^b x^2\,dy + \int_0^b y^2\,dy\right] dx.$$

$$= \int_{x=0}^{a}\left[x^2(y)_0^b + \left(y^3/3\right)_0^b\right] dx$$

$$= b\int_{x=0}^{a} x^2\,dx + \frac{b^3}{3}\int_{x=0}^{a} dx$$

$$= b\left[\frac{x^3}{3}\right]_0^a + \frac{b^3}{3}(x)_0^a$$

$$= \frac{ba^3}{3} + \frac{b^3}{3}.a$$

$$= \frac{ab}{3}(a^2+b^2)$$

$\therefore$ $$\int_0^a \int_0^b (x^2+y^2)\, dy.dx = \frac{ab}{3}(a^2+b^2)$$

Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dx.dy$

Sol:- The limits of the interior integration are functions x. Hence these are the limits of y.

$\therefore$ The outer limits 0, 1 are the limits of x. Hence we shall rewrite the given integral as $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx$

$$= \int_{x=0}^1 \left[\int_{y=x}^{\sqrt{x}} x^2\, dy + \int_{y=x}^{\sqrt{x}} y^2\, dy \right] dx$$

$$= \int_{x=0}^1 \left[x^2 (y)_x^{\sqrt{x}} + \left(\frac{y^3}{3}\right)_x^{\sqrt{x}} \right] dx$$

$$= \int_{x=0}^1 \left[x^2(\sqrt{x}-x) + \frac{1}{3}\left((\sqrt{x})^3 - x^3\right) \right] dx$$

$$= \int_{x=0}^1 \left(x^{5/2} - x^3 + \frac{1}{3}\left(x^{3/2} - x^3\right) \right) dx$$

$$= \int_{x=0}^1 x^{5/2}\, dx - \int_{x=0}^1 x^3\, dx + \frac{1}{3}\int_{x=0}^1 x^{3/2}\, dx - \frac{1}{3}\int_{x=0}^1 x^3\, dx$$

$$= \left(\frac{x^{7/2}}{7/2}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1 + \frac{1}{3}\left(\frac{x^{5/2}}{5/2}\right)_0^1 - \frac{1}{3}\left(\frac{x^4}{4}\right)_0^1$$

$$= \frac{2}{7}[1-0] - \frac{1}{4}[1-0] + \frac{1}{3}\cdot\frac{2}{5}[1-0] - \frac{1}{12}[1-0]$$

$$= \frac{2}{7} - \frac{1}{4} + \frac{2}{15} - \frac{1}{12}$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3}$$

$$= \frac{30+14-35}{105} = \frac{9}{105}$$

$$= \frac{3}{35}$$

$$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dx.dy = \frac{3}{35}$$

Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy.dx}{1+x^2+y^2}$

The given integral is $\displaystyle\int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{dy.dx}{(1+x^2)+y^2}$

put $p^2 = 1+x^2$

$p = \sqrt{1+x^2}$

$$= \int_{x=0}^{1} \int_{y=0}^{p} \frac{dy.dx}{p^2+y^2} \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\operatorname{Tan}^{-1}(x/a)\right]$$

$$= \int_{x=0}^{1} \left[\frac{1}{p}.\operatorname{Tan}^{-1}(y/p)_0^p\right] dx$$

$$= \int_{x=0}^{1} \frac{1}{p}\left[\operatorname{Tan}^{-1}(P/p) - \operatorname{Tan}^{-1}(0)\right] dx$$

$$= \int_{x=0}^{1} \frac{1}{p}\left[\operatorname{Tan}^{-1}(1) - \operatorname{Tan}^{-1}(0)\right] dx$$

$$= \int_{x=0}^{1} \frac{1}{p}\left[\pi/4 - 0\right] dx$$

$$\pi/4 \int_{x=0}^{1} \frac{dx}{\sqrt{1+x^2}} \qquad \left[\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x)\right]$$

$$= \pi/4 \cdot \left[\sinh^{-1}(x)\right]_0^1$$

$$= \pi/4 \ \sinh^{-1}(1)$$

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy.dx}{1+x^2+y^2} = \frac{\pi}{4} \sinh^{-1}(1)$$

Evaluate $\displaystyle\int_0^4 \int_{y^2/4}^{y} \frac{y}{x^2+y^2} \, dx.dy$

The given integral is $\displaystyle\int_{y=0}^{4} \int_{x=y^2/4}^{y} \cdot \frac{y}{x^2+y^2} \, dx \, dy$

$$= \int_{y=0}^{4} \left[y \cdot \int_{x=y^2/4}^{y} \frac{1}{x^2+y^2} \, dx \right] dy.$$

$$= \int_{y=0}^{4} \left[\cancel{y} \cdot \left(\frac{1}{\cancel{y}}\right) \left(\tan^{-1}(x/y)\right)_{y^2/4}^{y} \right] dy.$$

$$= \int_{y=0}^{4} \left[\tan^{-1}(y/y) - \tan^{-1}\left(y^2/4y\right)\right] dy$$

$$= \int_{y=0}^{4} \left[\tan^{-1}(1) - \tan^{-1}(y/4)\right] dy$$

$$= \int_{y=0}^{4} \pi/4 \, dy - \int_{y=0}^{4} \tan^{-1}(y/4) \, dy$$

$$\pi/4\,(y)_0^4 - \int_{y=0}^{4} \text{Tan}^{-1}(y/4)\,dy.$$

$$= \pi/4\,[4-0] - \int_{y=0}^{4} \text{Tan}^{-1}(y/4)\,dy.$$

$$= \pi - \left[\text{Tan}^{-1}(y/4)\int_0^4 dy\right] - \int_{y=0}^{4} \frac{1}{1+(y/4)^2}\cdot\frac{1}{4}\cdot y\,dy\Big].$$

$$= \pi - \left[\left(\text{Tan}^{-1} y/4 \cdot y\right)_0^4 - \int_0^4 \frac{16}{16+y^2}\cdot \cancel{\tfrac{y}{4}}\ \frac{y}{4}\cdot y\,dy\right]$$

$$= \pi - \left[4\,\text{Tan}^{-1}(1)\right] + \int_{y=0}^{4} \frac{4y}{16+y^2}\,dy.$$

$$= \pi - \cancel{4}(\pi/\cancel{4}) + 2\int_0^4 \frac{2y}{y^2+4^2}\,dy$$

$$\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right] \text{— formula}$$

$$= 2\left[\log(y^2+4^2)\right]_0^4$$

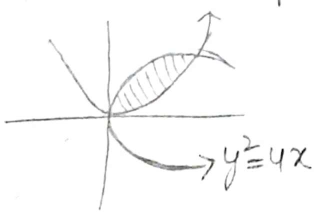
$$= 2\left[\log(32) - \log(16)\right]$$

$$= 2\log\left(\frac{\cancel{32}\,2}{\cancel{16}}\right)$$

$$= 2\log 2$$

$$\int_0^4 \int_{y^2/4}^{y} \frac{y}{x^2+y^2}\,dx.dy = 2\log 2.$$

Evaluate $\iint_R y.\,dx\,dy$. where R is the region bounded by the parabolas $y^2=4x$ and $x^2=4y$



Sol:- Given parabolas $y^2=4x$ —— ①

$x^2=4y$ —— ②

solving eq ① & ②, we get intersecting point

from ① $x = y^2/4$

put $x = y^2/4$ in ② we get

$$\left(\frac{y^2}{4}\right)^2 = 4y$$

$$\frac{y^4}{16} - 4y = 0$$

$$y\left(\frac{y^3}{16} - 4\right) = 0$$

$y = 0$ $\qquad \frac{y^3}{16} - 4 = 0$

$$y^3 = 64$$

$$y^3 = 4^3$$

$y = 0$ & $y = 4$

when $y = 0 \Rightarrow x = 0$.

$y = 4 \Rightarrow x = 4$.

The point of intersection is $(0, 0)$ and $(4, 4)$

for a fixed x, y is varying from $\frac{x^2}{4}$ to $2\sqrt{x}$

x varying from 0 to 4.

$$\iint_R y \, dx \, dy = \int_{x=0}^{4} \int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y . \, dy \, dx$$

$$= \int_{x=0}^{4} \int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y \cdot dy \cdot dx$$

$$= \int_{x=0}^{4} \left(\frac{y^2}{2}\right)_{x^2/4}^{2\sqrt{x}} dx$$

$$= \frac{1}{2}\int_{x=0}^{4}\left[(2\sqrt{x})^2 - \left(\frac{x^2}{4}\right)^2\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{4}\left(4x - \frac{x^4}{16}\right) dx$$

$$= \int_0^4 2x\,dx - \frac{1}{32}\int_0^4 x^4\,dx$$

$$= 2\left(\frac{x^2}{2}\right)_0^4 - \frac{1}{32}\left(\frac{x^5}{5}\right)_0^4$$

$$= 16 - \frac{1}{\cancel{32}}\left(\frac{\cancel{1024}^{\,32}}{5}\right)$$

$$= 16 - \frac{32}{5} = \frac{80-32}{5}$$

$$= 48/5$$

$$\iint_R y\,dx\,dy = 48/5$$

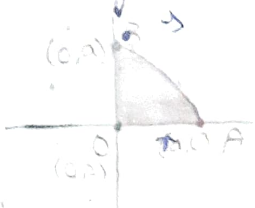
Evaluate $\iint_R xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$ — ①

$(x-h)^2+(y-k)^2=r^2$

$h\ k$ – centre r – radius

Let $x^2+y^2=a^2$ is a circle with centre $(0,0)$ and radius a units.

The given region R of integration is bounded by OABO



$\therefore$ for the point of integration

put $x=0$ ① we get $y=a$

$y=0$ we get $x=a$

The point of intersection is $(0,a)$ and $(a,0)$

for a fixed y we have x is varying from 0 to $\sqrt{a^2-y^2}$

$$\left(\because \begin{array}{l} x^2+y^2=a^2 \\ x^2=a^2-y^2 \\ x=\sqrt{a^2-y^2} \end{array}\right)$$

$\therefore$ y is varying from 0 to a.

$$\iint_R xy\,dx\,dy = \int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} xy\,dx\,dy$$

$$= \int_{y=0}^{a} y\cdot\left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a y\left(\sqrt{a^2-y^2}\right)^2 dy$$

$$= \frac{1}{2}\int_0^a y(a^2-y^2)\,dy$$

$$= \frac{1}{2}\int_0^a a^2y\,dy - \frac{1}{2}\int_0^a y^3\,dy$$

$$= \frac{1}{2}a^2\left(\frac{y^2}{2}\right)_0^a - \frac{1}{2}\left(\frac{y^4}{4}\right)_0^a$$

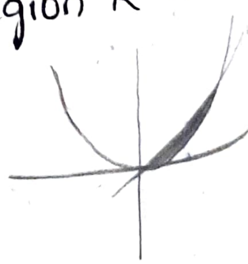
$$= \frac{1}{4}a^2(a^2) - \frac{1}{8}(a^4)$$

$$= \frac{a^4}{4} - \frac{a^4}{8}$$

$$= \frac{2a^4 - a^4}{8} = \frac{a^4}{8}.$$

$$\iint_R xy\,dx\,dy = \frac{a^4}{8}$$

Evaluate $\iint_R xy(x+y)\,dx\,dy$ over the region R bounded by $y = x^2$ — ① and $y = x$ — ②



Given curves $y = x^2$ — ①

$y = \sqrt{y}$ $\qquad y = x$ ~~$\sqrt{x^3}$~~ ②

put $y = x$ in ① we get

$$x=x^2$$
$$x-x^2=0$$
$$x(1-x)=0$$
$$x=0 \quad \text{and} \quad x=1$$

When $x=0 \Rightarrow y=0$ (from eq 2)

$x=1 \Rightarrow y=1$

$(0,0)$ and $(1,1)$ are the point of intersection of two curves

for a fixed x, y is varying from x^2 to x and x is varying from 0 to 1.

$$\iint_R xy(x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dy\cdot dx$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x^2y+xy^2)\,dy\cdot dx$$

$$= \int_{x=0}^{1}\left[x^2\int_{y=x^2}^{x} y\,dy + x\int_{y=x^2}^{x} y^2\,dy\right]dx$$

$$= \int_{x=0}^{1}\left[x^2\left(\frac{y^2}{2}\right)_{x^2}^{x} + x\left(\frac{y^3}{3}\right)_{x^2}^{x}\right]dx$$

$$= \int_{x=0}^{1}\left[\frac{x^2}{2}(x^2-x^4)+\frac{x}{3}(x^3-x^6)\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{1} x^4\,dx - \frac{1}{2}\int_{x=0}^{1} x^6\,dx + \frac{1}{3}\int_{0}^{1} x^4\,dx - \frac{1}{3}\int_{0}^{1} x^7\,dx$$

$$= \frac{1}{2}\left(\frac{x^5}{5}\right)_0^1 - \frac{1}{2}\left(\frac{x^7}{7}\right)_0^1 + \frac{1}{3}\left(\frac{x^5}{5}\right)_0^1 - \frac{1}{3}\left(\frac{x^8}{8}\right)_0^1$$

$$= \frac{1}{2}\frac{(1)}{5} - \frac{1}{14}(1) + \frac{1}{15}(1) - \frac{1}{24}(1)$$

$$= \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24}$$

$$= \frac{84 - 60 + 56 - 35}{840}$$

$$= \frac{140 - 95}{840} = \frac{45}{840} = \frac{9}{168} = \frac{3}{56}.$$

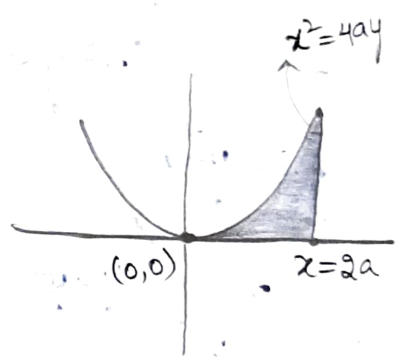
$$\iint_R xy(x+y)\,dx\,dy = \frac{3}{56}.$$

Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded X-axis ordinates $x=2a$ and the curve $x^2=4ay$.

Given that $y=0$ —— ①

$x=2a$ —— ②

$x^2=4ay$ —— ③



Using ② in ③ we get

$(2a)^2 = 4ay$

$\cancel{4}a^{\cancel{2}} = \cancel{4}\cancel{a}y$

$y=a$

When $x=2a \Rightarrow y=a$

$y=0 \Rightarrow x=0$

The point of intersection is $(2a, a)$ and $(0,0)$

→ For a fixed x, y is varying from 0 to $x^2/4a$. x is varying from 0 to $2a$

$$\iint_R xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dy\,dx$$

$$= \int_{x=0}^{2a} x \cdot \left(\frac{y^2}{2}\right)_0^{x^2/4a} dx$$

$$= \frac{1}{2} \int_0^{2a} x \left(\frac{x^4}{16a^2}\right) dx$$

$$= \frac{1}{32a^2} \int_{x=0}^{2a} x^5 \, dx$$

$$= \frac{1}{32a^2} \left(\frac{x^6}{6}\right)_0^{2a}$$

$$= \frac{1}{32a^2 (6)} \times (2a)^6$$

$$= \frac{1}{\cancel{32}\cancel{a^2} \times \cancel{6}_3} \times \cancel{a^6}^4 \times \cancel{64}^2$$

$$= \frac{a^4}{3}$$

$$\iint_R xy \, dx \, dy = \frac{a^4}{3}$$

Note :-

$$\int_0^{\pi/2} \sin^n\theta \, d\theta = \int_0^{\pi/2} \cos^n\theta \, d\theta$$

$$= \frac{n-1}{n} \times \frac{n-3}{n-2} \cdots\cdots \frac{1}{2} \cdot \pi/2 \text{ if } n \text{ is even}$$

$$= \frac{n-1}{n} \times \frac{n-3}{n-2} \cdots\cdots \frac{2}{3} \cdot 1 \text{ if } n \text{ is odd}$$

Find $\iint (x+y)^2 \, dx \, dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol:- Given, ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ——— ①

put $y = 0$

$$\frac{x^2}{a^2} = 1 \Rightarrow x^2 = a^2$$

$$x = \pm a$$

from ① $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$$y^2 = b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

The region of Integration is

$$-a \le x \le a, \quad -\frac{b}{a}\sqrt{a^2-x^2} < y < \frac{b}{a}\sqrt{a^2-x^2}$$

$$\iint_R (x+y)^2 dx\, dy = \int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{y=b/a\sqrt{a^2-x^2}} (x^2+y^2+2xy)\, dy . dx$$

$$= \int_{x=-a}^{a}\int_{y}^{b/a\sqrt{a^2-x^2}} (x^2+y^2) dx\, dy + \int_{x=-a}^{a}\int_{y}^{b/a\sqrt{a^2-x^2}} 2xy\, dy . dx$$

$\int_{-a}^{a} f(x)\, dx = 2\int_0^a$ ...
$= 0$ (o...

$$= 2\int_{x=-a}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}} (x^2+y^2) dy\, dx + 0.$$

$\left[\begin{array}{ll} x^2+y^2 & \text{is even} \\ xy & \text{is odd} \end{array}\right]$

$$= 2\int_{-a}^{a} x^2 (y)_0^{b/a} dx + 2\int_{-a}^{a} \left(\frac{y^3}{3}\right)_0^{b/a\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} x^2\left(b/a\sqrt{a^2-x^2}\right) dx + \frac{2}{3}\int_{-a}^{a} \frac{b^3}{a^3}(a^2-x^2)^{3/2} dx$$

$$= \frac{4b}{a}\int_0^a x^2\sqrt{a^2-x^2}\, dx + \frac{4b^3}{3a^3}\int_0^a (a^2-x^2)^{3/2} dx.$$

put $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

when $x=0 \Rightarrow \theta = 0$

$x = a \Rightarrow \theta = \pi/2$

$$= \frac{4b}{a}\int_0^{\pi/2} a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}\,.\,a\cos\theta\, d\theta + \frac{4b^3}{3a^3}\int_0^{\pi/2}(a^2-a^2\sin^2\theta)^{3/2}\, a\cos\theta\, d\theta$$

$$= 4a^3b\int_0^{\pi/2}\sin^2\theta\,.\,\cos^2\theta\, d\theta + \frac{4}{3}ab^3\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= a^3b\int_0^{\pi/2}(2\sin\theta\cos\theta)^2 d\theta + \frac{4}{3}ab^3\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= a^3b\int_0^{\pi/2}\sin^2 2\theta\, d\theta + \frac{4}{3}ab^3\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= a^3b\int_0^{\pi/2}\left(\frac{1-\cos4\theta}{2}\right)d\theta + \frac{4}{3}ab^3\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= \frac{a^3b}{2}\int_0^{\pi/2}d\theta - \frac{a^3b}{2}\int_0^{\pi/2}\cos4\theta\, d\theta + \frac{4}{3}ab^3\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= \frac{a^3b}{2}(\theta)_0^{\pi/2} - \frac{a^3b}{2}\left(\frac{\sin4\theta}{4}\right)_0^{\pi/2} + \frac{\cancel{4}}{\cancel{3}}ab^3\left[\frac{\cancel{3}}{\cancel{4}}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{a^3b}{2}[\pi/2] - 0 + \frac{\pi}{4}ab^3$$

$$= \frac{\pi}{4}a^3b + \frac{\pi}{4}ab^3$$

$$= \frac{\pi}{4}ab(a^2+b^2)$$

$$\iint (x+y)^2 dx\, dy = \frac{\pi}{4}ab(a^2+b^2)$$

Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the region bounded by the positive quadrant for which $x+y \le 1$.

Given that $x+y=1$

when $x=0 \Rightarrow y=1$

$y=0 \Rightarrow x=1$

point of intersection $(0,1)$ and $(1,0)$

Limits :- x varying from 0 to 1.

y - varying from 0 to $1-x$.

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} (x^2+y^2)\,dy\,dx.$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{1-x} x^2\,dy + \int_{y=0}^{1-x} y^2\,dy\right]dx$$

$$= \int_{x=0}^{1} x^2 (y)_0^{1-x}\,dx + \int_{x=0}^{1}\left(\frac{y^3}{3}\right)_0^{1-x} dx$$

$$= \int_{x=0}^{1} x^2(1-x)\,dx + \frac{1}{3}\int_{x=0}^{1}(1-x)^3\,dx$$

$$= \int_{x=0}^{1} x^2\,dx - \int_{x=0}^{1} x^3\,dx + \frac{1}{3}\int_{x=0}^{1}(1-x)^3\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1 + \frac{1}{3}\left[\frac{(1-x)^4}{4(-1)}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} + \left(\frac{-1}{12}\right)[0-1]$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{4-3+1}{12}$$

$$= 2/12 = 1/6$$

$$\iint_R (x^2+y^2)\,dx\,dy = 1/6$$

Evaluate $\iint_R y \cdot dx\, dy$ where R is the region bounded by y-axis and the curve $y = x^2$ & the line $x+y=2$ in first quadrant.

Given that $x = 0$ —— ①

$y = x^2$ —— ②

$x + y = 2$ —— ③

using ② in ③ we get

$$x + x^2 = 2$$

$$x^2 + x - 2 = 0$$

$$x^2 + 2x - x - 2 = 0$$

$$x(x+2) - 1(x+2) = 0$$

$$(x-1)(x+2) = 0$$

$$x = -2, 1$$

$$\boxed{x = 1}$$

when $x = 1 \Rightarrow y = 1$

when $x = 0 \Rightarrow y = 0$

$x = 0 \Rightarrow y = 2$.

The point of intersection is $(1,1), (0,0), (0,2)$

x is varying from 0 to 1

y is varying from x^2 to $2-x$

$$\iint_R y \cdot dx\, dy = \int_{x=0}^{1} \int_{y=x^2}^{2-x} y \cdot dy \cdot dx$$

$$= \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_{x^2}^{2-x} dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left((2-x)^2\right) - (x^2)^2 \, dx$$

$$= \frac{1}{2}\left[\int_{x=0}^{1} (2-x^2)^2 dx - \int_{x=0}^{1} x^4 dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{(2-x)^3}{3(-1)}\right)_0^1 - \left(\frac{x^5}{5}\right)_0^1\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{-3}\left((1)-(8)\right)\right) - \frac{1}{5}\right]$$

$$= \frac{1}{2}\left[\frac{-7}{-3} - \frac{1}{5}\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} - \frac{1}{5}\right] = \frac{1}{2}\left[\frac{35-3}{15}\right]$$

$$= \frac{1}{\cancel{2}} \times \frac{\overset{16}{\cancel{32}}}{15}$$

$$= \frac{16}{15}.$$

Evaluate $\iint (x+y)\, dx dy$ over the region in the positive qua[drant] bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} = 1 \Rightarrow x^2 = a^2$$

$\boxed{x = a}$ ($\because$ since +ve quadrant)

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

x is varying from 0 to a

y is varying from 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

$$\iint_R (x+y)\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}} (x+y)\,dy\cdot dx$$

$$= \int_{x=0}^{a} x\cdot (y)_0^{b/a\sqrt{a^2-x^2}}\,dx + \int_{x=0}^{a} \left(\frac{y^2}{2}\right)^{b/a\sqrt{a^2-x^2}} dx$$

$$= \frac{b}{a}\int_0^a x\sqrt{a^2-x^2}\,dx + \frac{b^2}{2a^2}\int_0^a (a^2-x^2)\,dx$$

put $a^2-x^2=t^2$

$$-\cancel{2}x\,dx = \cancel{2}t\,dt$$

$$x\,dx = -t\,dt$$

when $x=0 \Rightarrow t=a$

$x=a \Rightarrow t=0$

$$= \frac{b}{a}\int_0^a \sqrt{t^2}\,(-t\,dt) + \frac{b^2}{2a^2}\left[\int_0^a a^2\,dx - \int_0^a x^2\,dx\right]$$

$$= \frac{b}{a}\int_0^a t^2\,.\,dt + \frac{b^2}{2a^2}\left[\int_0^a a^2 dx - \int_0^a x^2 dx\right]$$

$$= \frac{b}{a}\left(\frac{t^3}{3}\right)_0^a + \frac{b^2}{2a^2}\left[a^2(x)_0^a - \left(\frac{x^3}{3}\right)_0^a\right]$$

$$= \frac{b}{3a}\left[a^3\right] + \frac{b^2}{2a^2}\left[a^2(a) - \frac{a^3}{3}\right]$$

$$= \frac{ba^2}{3} + \frac{ab^2}{2} - \frac{ab^2}{6}$$

$$= \frac{ba^2}{3} + \frac{3ab^2-ab^2}{6}$$

$$= \frac{ba^2}{3} + \frac{\cancel{2}ab^2}{\cancel{6}\,3}$$

$\frac{1}{3}(ab^2+ba^2)$

$\iint (x+y)\,dx\,dy = \frac{1}{3}(ab^2+ba^2)$

Double integral in polar Co-Ordinates:

Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r.\,dr.\,d\theta$

Sol:-

$$\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta = \int_{\theta=0}^{\pi}\left[\int_{r=0}^{a\sin\theta} r\,dr\right]d\theta$$

$$= \int_{\theta=0}^{\pi}\left(\frac{r^2}{2}\right)_0^{a\sin\theta} . d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta.\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi} a^2\left[\frac{1}{2}(1-\cos2\theta)d\theta\right]$$

$$\left[\begin{aligned}\cos2\theta &= 1-2\sin^2\theta\\ 2\sin^2\theta &= 1-\cos2\theta\\ \sin^2\theta &= \tfrac{1}{2}(1-\cos2\theta)\end{aligned}\right.$$

$$= \frac{a^2}{4}\int_{\theta=0}^{\pi}(1-\cos2\theta)d\theta$$

$$= \frac{a^2}{4}\int_{\theta=0}^{\pi} d\theta - \frac{a^2}{4}\int_{\theta=0}^{\pi}\cos2\theta\,d\theta$$

$$= \frac{a^2}{4}(\theta)_0^{\pi} - \left(\frac{\sin2\theta}{2}\right)_0^{\pi}$$

$$= \frac{a^2}{4}(\pi)$$

$$\int_0^{\pi}\int_0^{a\sin\theta} r.\,dr.\,d\theta = \frac{\pi a^2}{4}$$

Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,.dr\,d\theta$

$$\int_0^{\pi}\int_0^{a(1+\cos\theta)} r.\,dr.\,d\theta = \int_0^{\pi}\left[\int_{r=0}^{a(1+\cos\theta)} r\ dr\right]d\theta$$

$$= \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} .\ d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2.\,d\theta$$

$\left[\begin{array}{l}\cos 2\theta = 2\cos^2\theta - 1\\ 2\cos^2\theta = 1+\cos 2\theta\\ 1+\cos\theta = 2\cos^2\theta/2\end{array}\right]$

$$= \frac{1}{2}\int_0^{\pi} a^2(2\cos^2\theta/2)^2 d\theta.$$

$$= 2a^2\int_{\theta=0}^{\pi} \cos^4\theta/2.\ d\theta$$

put $\theta/2 = t$

$\theta = 2t.$

$d\theta = 2dt$

when $\theta=0 \Longrightarrow t=0$

$\theta=\pi \Longrightarrow t=\pi/2$

$$= 2a^2\int_0^{\pi/2} \cos^4 t\, 2dt$$

$$= 4a^2\int_0^{\pi/2} \cos^4 t\,.\,dt$$

$\int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{n-1}{n}\times\frac{n-3}{n-2}\times\cdots\frac{1}{2}\frac{\pi}{2}$

$$= 4a^2\left[\frac{4-1}{4}.\ \frac{4-3}{4-2}.\ \frac{\pi}{2}\right]$$

$$= 4a^2\left[\frac{3}{4}.\ \frac{1}{2}.\ \frac{\pi}{2}\right]$$

$$= \frac{3a^2\pi}{4}$$

$$\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\ dr.\,d\theta = \frac{3a^2\pi}{4}$$

Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\, dr.d\theta$

Given integral can be written as:

C $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\, dr\, d\theta = \int_0^{\pi/4}\left[\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\, dr\right] d\theta$

E

$\underline{\underline{sol}}$

put $a^2-r^2=t^2$

$-\cancel{2}r\,dr = \cancel{2}t\,dt$

$r\,dr = -t.dt$

when $r=0 \Rightarrow t=a$

$r=a\sin\theta \Rightarrow t=a\cos\theta\, d\theta$

$$= \int_{\theta=0}^{\pi/4}\left[\int_{t=a}^{a\cos\theta} \frac{-t.dt}{\sqrt{t^2}}\right] d\theta$$

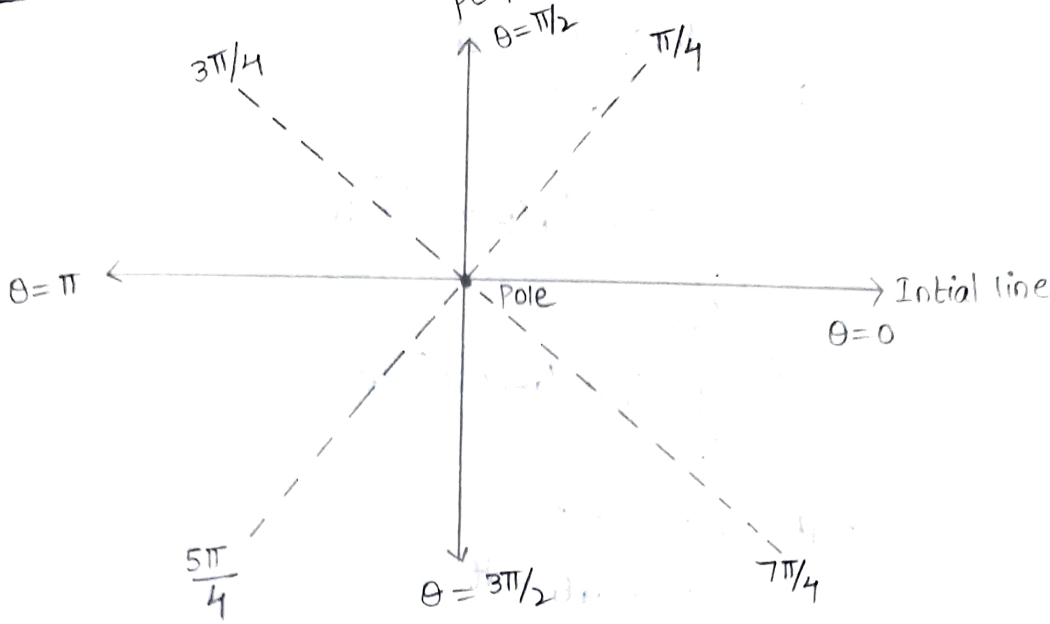
$$= -\int_{\theta=0}^{\pi/4} (t)_a^{a\cos\theta}\, d\theta$$

$$= -\int_{\theta=0}^{\pi/4} (a\cos\theta - a)\, d\theta.$$

$$= -a\int_0^{\pi/4} \cos\theta\, d\theta + a\int_0^{\pi/4} d\theta$$

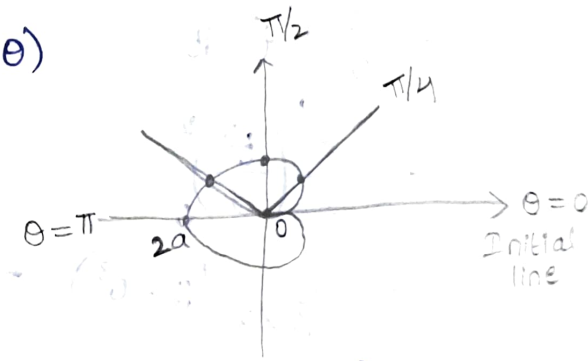
$$= -a(\sin\theta)_0^{\pi/4} + a(\theta)_0^{\pi/4}$$

Evaluation of polar limits :-



Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cordioid $r=a(1-\cos\theta)$ above the initial line.

Given, curve is $r=a(1-\cos\theta)$



The given curve $r=a(1-\cos\theta)$ is symmetrical about the intial line and passing through the pole.

$r=a(1-\cos\theta)$

θ	0	$\pi/4$	$\pi/2$	π
r	0	$a(1-\frac{1}{\sqrt{2}})$	a	$2a$

The region of integration R above the intial line is the shaded area in the diagram

$\therefore$ R is varying from 0 to $a(1-\cos\theta)$ and θ is varying from 0 to π.

$$\therefore \iint r\sin\theta\, dr\, d\theta = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1-\cos\theta)} r\sin\theta\, dr.\, d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left[\int_{r=0}^{a(1-\cos\theta)} r\, dr\right] . d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \left[a(1-\cos\theta)\right]^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin\theta\,(1-\cos\theta)^2\, d\theta$$

put $1-\cos\theta = t$

$\sin\theta\, d\theta = dt$

when $\theta = 0 \Rightarrow t = 0$

$\theta = \pi \Rightarrow t = 2$

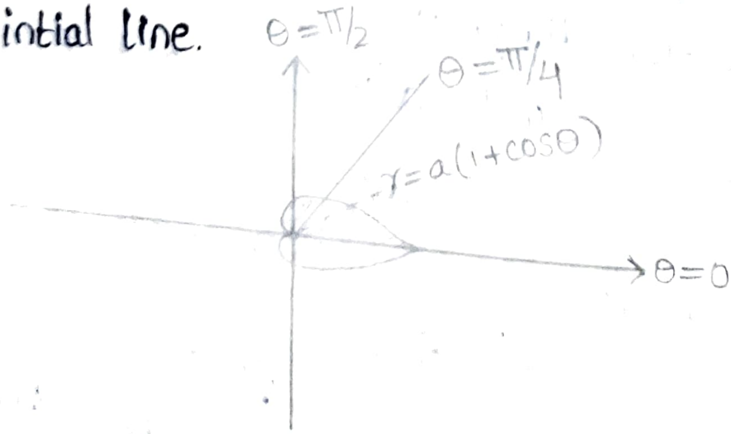
$$= \frac{a^2}{2}\int_0^{2} t^2\, dt$$

$$= \frac{a^2}{2}\left(\frac{t^3}{3}\right)_0^2$$

$$= \frac{a^2}{2\times 3}(2^3 - 0^3) = \frac{a^2}{\cancel{6}\,3} \times \cancel{8}\,4.$$

$$\iint r\sin\theta\, dr\, d\theta = \frac{4a^2}{3}.$$

Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cordiod $r = a(1+\cos\theta)$ above the intial line.



θ	0	$\pi/4$	$\pi/2$	π
r	$2a$	$a(1+\frac{1}{\sqrt{2}})$	a	0

The given curve $r = a(1+\cos\theta)$ is symmetrical about the intial line and passing through the pole.

The region of Integration R above the intial line is the shaded area in the diagram.

R is varying from 0 to $a(1+\cos\theta)$ and θ is varying from 0 to π.

$$\iint r\sin\theta \, dr\, d\theta = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r\sin\theta \, dr. d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left[\int_{r=0}^{a(1+\cos\theta)} r\, dr \right] d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta.$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta\left(a^2(1+\cos\theta)^2\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin\theta\,(1+\cos\theta)^2 d\theta$$

put $1+\cos\theta = t$

$-\sin\theta\, d\theta = dt$

$\sin\theta\, d\theta = -dt.$

when

$\theta = 0 \Rightarrow t = 2$

$\theta = \pi \Rightarrow t = 0$

$$= \frac{a^2}{2}\int_{t=2}^{0} t^2(-dt)$$

$$= \frac{a^2}{2}\int_0^2 t^2. dt = \frac{a^2}{2}\left(\frac{t^3}{3}\right)_0^2$$

$$= \frac{a^2}{\cancel{2}\times 3}(\cancel{8})^4 = \frac{4a^2}{3}.$$

$$\iint r\sin\theta\, dr\, d\theta = \frac{4a^2}{3}.$$